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A dynamic multiscale lifting computation method using Daubechies wavelet

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Abstract

An important property of wavelet multiresolution analysis is the capability to represent functions in a dynamic multiscale manner, so the solution in the wavelet domain enables a hierarchical approximation to the exact solution. The typical problem that arises when using Daubechies wavelets in numerical analysis, especially in finite element analysis, is how to calculate the connection coefficients, an integral of products of wavelet scaling functions or derivative operators associated with these. The method to calculate multiscale connection coefficients for stiffness matrices and load vectors is presented for the first time. And the algorithm of multiscale lifting computation is developed. The numerical examples are given to verify the effectiveness of such a method.

Keywords: Daubechies Wavelet; Multiscale; Connection coefficients

1. Introduction

Wavelet multiresolution theory [25] provides a powerful mathematical tool for function approximation and multiscale representations. Typical applications of wavelet analysis include signal processing, numerical analysis, and large-scale computation. An important property of wavelet analysis is the capability to represent function in a dynamic multiscale manner, so solution in the wavelet domain enables a hierarchical approximation to the approach solution rapidly. Such dynamic multiscale

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properties make wavelets strongly attractive for the numerical solution of partial differential equations (PDEs).

Dahmenen [12] has reviewed the recent developments of wavelet-based schemes for PDEs and proved the key features of multiscale wavelet bases; cancellation property and norm equivalences are essential for the numerical treatment of operator equations. The first feature accounts for efficient matrix multiplication and the second one guarantees optimal preconditioning. Amaratunga [2] has presented wavelet-Galerkin solution for the one-dimensional Helmholtz boundary value problem with periodic boundary condition, which is superior to the finite difference method. Ho [16] has presented a new wavelet-Galerkin formulation to solve parabolic equations in finite domains based on weak form functionals. Bertoluzza [3] has proved the convergence of an adaptive wavelet algorithm for solution of elliptic PDEs. Al-Qassab [1] has used the wavelet to study the free vibrations of an elastic cable and obtained good results in the singularity region with the feature of wavelets. Jang [17] has recently developed an adaptive wavelet-Galerkin method for two-dimensional elliptic problems defined in general domains.

In the current literature, Daubechies compactly supported orthogonal wavelet is widely used. Because Daubechies wavelet functions provide the stable bases for $L^2(R)$, the vector space of measurable and square-integrable one-dimensional functions, the combination of such multiscale bases with finite element method (FEM) leads to adaptive refinement strategies, which are guaranteed to converge in a wide range of cases. Several cases have been studied where such bases are available and have proved to offer significant advantages. For instance, Jaffard [18] has already proved that the condition number of the stiffness matrix is independent of mesh size in the wavelet basis. Ko [20,21] has presented the finite element formulated in terms of wavelet basis functions, and the approximation properties of the Daubechies wavelet-based elements are verified through a Neumann problem. Christon [10] has developed a multiscale linear finite element based on a wavelet hierarchical change-of-basis. Castro [7] has used Daubechies wavelet base to implement the stress model of hybrid-mixed stress finite elements. By adopting the scaling functions of Daubechies wavelets as interpolating function, the one-dimensional [24] and two-dimensional [9] wavelet finite elements are constructed, and the overall performances of the proposed wavelet elements are evaluated using the well-established finite element testing problems.

The typical problem that arises when using Daubechies wavelets in numerical analysis, especially in finite element analysis, is how to calculate the connection coefficients, an integral of products of wavelet scaling functions or derivative operators associated with these. Beylkin [4], Dahmen [13] and Wang [28] have presented the algorithm for computing the connection coefficients, such as

$$ {}^{1}\Gamma_{k,l}^{m,n} = \int_{-\infty}^{+\infty} \phi^{(m)}(\xi - k)\phi^{(n)}(\xi - l)\,\mathrm{d}\xi, \tag{1} $$

where $k, l \in Z$, $\phi(x)$ denotes the scaling function of Daubechies wavelets, and the superscripts m and n refer to differentiation. Such a calculation on unbounded domains is limited to cases where the problem is unbounded or the boundary condition is periodic. In order to apply the wavelet-Galerkin method to the solution of finite domain problems, Chen [8], Yang [30] and Monasse [26] have given the integral on the interval $[0, x]$, such as

$$ {}^{2}\Gamma_{k}^{m,n} = \int_{0}^{x} \phi^{(m)}(\xi - k)\phi^{(n)}(\xi)\,\mathrm{d}\xi. \tag{2} $$

Table 1
The four types of integrals

Type of integrals	The lower bound	The upper bound	Scale
(1)	$-\infty$	$+\infty$	0
(2)	0	x	0
(3)	0	2^j	0
(4)	0	1	0

While Lin [23] uses wavelet time difference scheme to solve Burgers equation, the integral is proposed such as

$$ {}^3\Gamma_{k,l}^{m,n} = \int_0^{2^j} \phi^{(m)}(\xi - k)\phi^{(n)}(\xi - l)\,\mathrm{d}\xi, \tag{3} $$

where j denotes the scale of wavelets. In order to construct wavelet finite element, Ko [20] and Ma [24] have studied the integral on the interval [0, 1], such as

$$ {}^4\Gamma_{k,l}^{m,n} = \int_0^{1} \phi^{(m)}(\xi - k)\phi^{(n)}(\xi - l)\,\mathrm{d}\xi. \tag{4} $$

The four types of integrals are listed in Table 1. We can see that all the above connection coefficients are calculated on scale 0 ($j = 0$). They are only different in the lower or upper integral bounds.

In order to completely use the interesting multiresolution properties of wavelets to construct the dynamic multiscale algorithm, the key problem is how to calculate connection coefficients at different resolution j (j=0, 1, 2, ...). We will focus on the details of how to calculate these multiscale connection coefficients. To our knowledge, the results of this paper offer details for the first time, and this algorithm helps us obtaining the adaptive multiscale lifting computing scheme. Firstly, we give the brief description of multiresolution analysis and Daubechies wavelets.

2. Multiresolution analysis and Daubechies wavelets

A multiresolution analysis of $L^2(R)$ is defined as a sequence of nested subspaces V_j with scaling function $\phi(x)$ if the following properties hold [25]:

$$ \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots, \tag{5} $$

$$ \mathrm{clos}_{L^2}\left(\bigcup_{j\in Z} V_j\right) = L^2(R), \tag{6} $$

$$ \bigcap_{j\in Z} V_j = \{0\}, \tag{7} $$

$$ f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad j \in Z. \tag{8} $$

The V_j's are called approximation spaces. The scaling function $\phi(x)$ belongs to $\boldsymbol{V}_0$ and the set of $\{\phi(x-k) \,|\, k \in Z\}$ is a Riesz basis of $\boldsymbol{V}_0$. A sequence $\mathbf{p}_k \in \ell^2(Z)$ exists, $\ell^2(Z)$ denotes the integer space of all square-summable bi-infinite sequences, such that the scaling function $\phi(x)$ satisfies a refinement equation

$$\phi(x) = \sum_k p_k \phi(2x-k), \quad k \in Z, \tag{9}$$

which is also called two-scale relation. Through dilation and translation, the set of functions $\{\phi_{j,k}(x) \,|k \in Z\}$ with

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \tag{10}$$

is a Riesz basis of $\boldsymbol{V}_j$.

Multiresolution property means that $\boldsymbol{V}_j$ is a subset of $\boldsymbol{V}_{j+1}$. So each element of $\boldsymbol{V}_{j+1}$ can be uniquely written as the orthogonal sum of an element in $\boldsymbol{V}_j$ and an element in $\boldsymbol{W}_j$ that contains the complementing details, i.e.,

$$\boldsymbol{V}_{j+1} = \boldsymbol{V}_j \oplus \boldsymbol{W}_j. \tag{11}$$

Let W_j be the span of $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, which is called wavelet function. A sequence $\mathbf{q}_k \in \ell^2(Z)$ exists such that the wavelet function $\psi(x)$ satisfies

$$\psi(x) = \sum_k q_k \phi(2x-k), \quad k \in Z. \tag{12}$$

As an example of multiresolution analysis, a family of orthogonal Daubechies wavelets with compactly supported property has been constructed by Daubechies in [15]. The main properties of Daubechies wavelets are summarized in the appendix.

Due to the lack of the explicit Daubechies scaling function expression, the connection coefficients are difficult to calculate.

3. Connection coefficients

While using the scaling function of Daubechies wavelet as a test function of finite element method, we would obtain two typical connection coefficients to form stiffness matrices and load vectors [9], such as

$$\Lambda_{N,k,l}^{j,m,n} = \int_{-\infty}^{+\infty} \chi_{[0,1]}(\xi)\phi_N^{(m)}(2^j\xi - k)\phi_N^{(n)}(2^j\xi - l)\,\mathrm{d}\xi, \tag{13}$$

$$R_{N,k}^{j,m} = \int_{-\infty}^{+\infty} \chi_{[0,1]}(\xi)\xi^m \phi_N(2^j\xi - k)\,\mathrm{d}\xi, \tag{14}$$

where

$$\chi_{[0,1]}(\xi) = \begin{cases} 1 & 0 \leqslant \xi \leqslant 1, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

which satisfies a simple two-scale relation

$$\chi_{[0,1]}\left(\frac{1}{2}\zeta\right)=\chi_{[0,1]}(\zeta)+\chi_{[1,2]}(\zeta)=\chi_{[0,1]}(\zeta)+\chi_{[0,1]}(\zeta-1). \tag{16}$$

This section will give the details on how to calculate multiscale connection coefficients.

3.1. Connection coefficients for stiffness matrices

Using two-scale relation Eq. (9), we obtain

$$\phi_N(2^j x-k)=\sum_s p_s\phi_N(2^{j+1}x-2k-s). \tag{17}$$

Differentiating the above equation m times, we have

$$2^{jm}\phi_N^{(m)}(2^j x-k)=\sum_s p_s 2^{(j+1)m}\phi_N^{(m)}(2^{j+1}x-2k-s). \tag{18}$$

Substituting Eqs. (16) and (18) into Eq. (13), we have

$$\begin{aligned}\Lambda_{N,k,l}^{j,m,n}&=\int_{-\infty}^{+\infty}\chi_{[0,1]}(\xi)\phi_N^{(m)}(2^j\xi-k)\phi_N^{(n)}(2^j\xi-l)\,\mathrm{d}\xi\\&=\int_{-\infty}^{+\infty}\chi_{[0,1]}(\xi)2^m\sum_s p_s\phi_N^{(m)}(2^{j+1}\xi-2k-s)2^n\sum_t p_t\phi_N^{(n)}(2^{j+1}\xi-2l-t)\,\mathrm{d}\xi.\end{aligned} \tag{19}$$

Supposing $\gamma=2\xi$,

$$\begin{aligned}\Lambda_{N,k,l}^{j,m,n}&=2^{m+n-1}\int_{-\infty}^{+\infty}\chi_{[0,1]}\left(\frac{\gamma}{2}\right)\sum_s p_s\phi_N^{(m)}(2^j\gamma-2k-s)\sum_t p_t\phi_N^{(n)}(2^j\gamma-2l-t)\,\mathrm{d}\gamma\\&=2^{m+n-1}\int_{-\infty}^{+\infty}\{\chi_{[0,1]}(\gamma)+\chi_{[0,1]}(\gamma-1)\}\sum_s p_s\phi_N^{(m)}(2^j\gamma-2k-s)\\&\quad\times\sum_t p_t\phi_N^{(n)}(2^j\gamma-2l-t)\,\mathrm{d}\gamma\\&=2^{m+n-1}\int_{-\infty}^{+\infty}\left\{\chi_{[0,1]}(\gamma)\sum_{s,t}p_s p_t\phi_N^{(m)}(2^j\gamma-2k-s)\phi_N^{(n)}(2^j\gamma-2l-t)\right.\\&\quad\left.+\chi_{[0,1]}(\gamma-1)\sum_{s,t}p_s p_t\phi_N^{(m)}(2^j\gamma-2k-s)\phi_N^{(n)}(2^j\gamma-2l-t)\right\}\mathrm{d}\gamma\\&=2^{m+n-1}\sum_{s,t}p_s p_t(\Lambda_{N,2k+s,2l+t}^{j,m,n}+\Lambda_{N,2k+s-2^j,2l+t-2^j}^{j,m,n}).\end{aligned} \tag{20}$$

The above equation can be written as

$$\Lambda_{N,k,l}^{j,m,n} = 2^{m+n-1} \sum_{s,t} (p_{s-2k} p_{t-2l} + p_{s-2k+2^j} p_{t-2l+2^j}) \Lambda_{N,s,t}^{j,m,n}, \tag{21}$$

where $-(2N-2) \leqslant k, l \leqslant 2^j - 1$. Expressing in matrix form

$$(2^{m+n-1}\mathbf{A} - \mathbf{I})\mathbf{\Lambda}_N^{j,m,n} = 0, \tag{22}$$

where $\mathbf{A}$ is the coefficients matrix of Eq. (21), $\mathbf{I}$ is an identity matrix, and $\mathbf{\Lambda}_N^{j,m,n}$ is the $(2^j + 2N - 1)x(2^j + 2N - 1)$ stiffness matrix. The matrix $\mathbf{\Lambda}_N^{j,m,n}$ cannot be determined uniquely through the homogeneous Eq. (22), so independent inhomogeneous equations are required for unique solution. An important additional property of Daubechies scaling function $\phi_N(x)$ is that low-order polynomials can be expressed as a linear combination of its translates [6],

$$x^q = \sum_k c_{j,k}^q \phi_N(2^j x - k) \quad q \leqslant N - 1, \tag{23}$$

where

$$c_{j,k}^q = \langle x^q, \phi_N(2^j x - k) \rangle. \tag{24}$$

Differentiating Eq. (23) m times,

$$q(q-1)\cdots(q-m+1)x^{q-m} = 2^{jm} \sum_k c_{j,k}^q \phi_N^{(m)}(2^j x - k). \tag{25}$$

Similarly, differentiating n times the expanding equation of x^w as Eq. (23), then multiplying both sides and taking integration from 0 to 1, we have

$$\int_0^1 \frac{q!}{(q-m)!} \frac{w!}{(w-n)!} x^{q+w-m-n} \, \mathrm{d}x = 2^{j(m+n)} \sum_{k,l} c_{j,k}^q c_{j,l}^w \int_0^1 \phi_N^{(m)}(2^j x - k) \phi_N^{(n)}(2^j x - l) \, \mathrm{d}x, \tag{26}$$

or

$$\frac{q!}{(q-m)!} \frac{w!}{(w-n)!} \frac{2}{q+w-m-n+1} = 2^{j(m+n)} \sum_{k,l} c_{j,k}^q c_{j,l}^w \Lambda_{N,k,l}^{j,m,n} \tag{27}$$

Combining Eqs. (27) with (22), we can obtain the connection coefficients $\mathbf{\Lambda}_N^{j,m,n}$ for stiffness matrices. For 3-order Daubechies wavelet, the connection coefficients $\mathbf{\Lambda}_3^{0,1,1}$ and $\mathbf{\Lambda}_3^{1,1,1}$ are listed in Tables 2 and 3, from which we can see the matrices are symmetric.

Table 2
Connection coefficients for $\mathbf{\Lambda}_3^{0,1,1}$

1.9836E−4	1.5308E−4	−4.7795E−3	8.4075E−3	−5.3571E−3
	4.6024E−4	−1.2830E−1	2.0344E−1	−1.2269E−1
		4.6736E−1	−1.0118	6.7753E−1
			3.0519	−2.2519
				1.7024

Table 3
Connection coefficients for $\mathbf{\Lambda}_3^{1,1,1}$

3.9672E−4	3.0616E−3	−9.5589E−3	1.6815E−2	−1.0714E−2	3.3964E−16
	9.2444E−2	−2.5354E−1	3.9732E−1	−2.2857E−1	−1.0714E−2
		1.0268	−2.2802	1.7619	−2.4539E−1
			7.0385	−6.5274	1.3551
				9.5085	−4.5038
					3.4048

3.2. Connection coefficients for load vectors

In order to calculate the connection coefficients of Eq. (14), we solve it for $m=0$, firstly. Similarly supposing $\gamma = 2\xi$,

$$\begin{aligned} R_{N,k}^{j,0} &= \int_{-\infty}^{+\infty} \chi_{[0,1]}(\xi)\xi^0\phi_N(2^j\xi - k)\,\mathrm{d}\xi = \sum_i p_i \int_{-\infty}^{+\infty} \chi_{[0,1]}(\xi)\phi_N(2^{j+1}\xi - 2k - i)\,\mathrm{d}\xi \\ &= \frac{1}{2}\sum_i p_i \int_{-\infty}^{+\infty} [\chi_{[0,1]}(\gamma) + \chi_{[0,1]}(\gamma - 1)]\phi_N(2^j\gamma - 2k - i)\,\mathrm{d}\gamma \\ &= \frac{1}{2}\sum_i p_i(R_{2k+i}^{j,0} + R_{2k+i-2^j}^{j,0}), \end{aligned} \tag{28}$$

where $-(2N-2)\leqslant k\leqslant 2^j - 1$. We obtain

$$R_{N,k}^{j,0} = \frac{1}{2}\sum_i (P_{i-2k} + P_{i-2k+2^j})R_{N,i}^{j,0}. \tag{29}$$

Also the additional inhomogeneous equation for a unique solution is required. Integrating Eq. (23)

$$\int_0^1 x^q\,\mathrm{d}x = \sum_k c_{j,k}^q \int_0^1 \phi_N(2^j x - k)\,\mathrm{d}x \quad q\leqslant N-1, \tag{30}$$

we obtain

$$\frac{1}{q+1} = \sum_k c_{j,k}^q R_{N,k}^{j,0} \quad q\leqslant N-1. \tag{31}$$

Table 4
Connection coefficients for load vectors

$\mathbf{R}_3^{0,1}$	$\mathbf{R}_3^{1,1}$
2.7180E−6	6.7950E−7
−1.1880E−4	5.6207E−5
−1.7774E−2	−9.3021E−4
1.1126E−1	−4.4344E−3
4.0663E−1	2.5347E−1
	2.5184E−1

Supplying Eq. (29) with the above equation, we can solve $R_{N,k}^{j,0}$. Then we can calculate the connection coefficients (14) for $m > 0$,

$$\begin{aligned} R_{N,k}^{j,m} &= \int_0^1 \xi^m \phi_N(2^j \xi - k)\,\mathrm{d}\xi = \sum_i p_i \int_{-\infty}^{+\infty} \chi_{[0,1]}(\xi)\xi^m \phi_N(2^{j+1}\xi - 2k - i)\,\mathrm{d}\xi \\ &= \frac{1}{2^{m+1}} \sum_i p_i \int_{-\infty}^{+\infty} \gamma^m [\chi_{[0,1]}(\gamma) + \chi_{[0,1]}(\gamma - 1)]\phi_N(2^j\gamma - 2k - i)\,\mathrm{d}\gamma \\ &= \frac{1}{2^{m+1}} \sum_i p_i \left[R_{N,2k+i}^{m} + \int_{-\infty}^{+\infty} \gamma^m \chi_{[0,1]}(\gamma - 1)\phi_N(2^j\gamma - 2k - i)\,\mathrm{d}\gamma \right] \\ &= \frac{1}{2^{m+1}} \sum_i p_i \left[R_{N,2k+i}^{m} + \int_{-\infty}^{+\infty} \chi_{[0,1]}(\gamma)(\gamma + 1)^m \phi_N(2^j\gamma + 2^j - 2k - i)\,\mathrm{d}\gamma \right] \\ &= \frac{1}{2^{m+1}} \sum_i p_i \left(R_{N,2k+i}^{m} + \sum_{s=0}^{m} \binom{m}{s} R_{N,2k+i-2^j}^{m-s} \right), \end{aligned} \tag{32}$$

or

$$(2^{m+1}\mathbf{I} - \mathbf{B}) R_{N,k}^{j,m} = \sum_i P_{i-2k+2^j} \sum_{s=1}^{m} \binom{m}{s} R_{N,i}^{m-s}, \tag{33}$$

where

$$\mathbf{B} = \sum_{i,k} (P_{i-2k} + P_{i-2k+2^j}). \tag{34}$$

Solving Eq. (33) we can obtain the load vector $\mathbf{R}_N^{j,m}$. For 3-order Daubechies wavelets, the connection coefficients $\mathbf{R}_3^{0,1}$ and $\mathbf{R}_3^{1,1}$ are listed in Table 4.

4. Multiscale lifting computation

The multiresolution property of wavelets indicates that, as j gets larger, more information is revealed by the approximation. Such multiscale schemes hinge upon making successive corrections of current

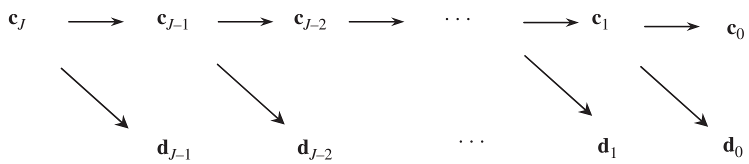

Fig. 1. Wavelet decomposition.

solutions when progressing to finer scales of discretization. Such an ability to represent functions at different levels of resolution lays a foundation for multiscale lifting numerical algorithm based on wavelet analysis.

Eq. (6) means that each function $f(x)$ in space $L^2(R)$ can be approximated with the projection $P_j f(x)$ in V_j, and the projection eventually captures all details or information of the initial function $f(x)$ as scale j gets larger (i.e., as $j \to +\infty$) [5], such as

$$\lim_{j\to+\infty} \|f(x) - P_j f(x)\| = 0. \tag{35}$$

Meanwhile Eq. (7) means that with decreasing scale j the projection $P_j f(x)$ can become very small and all details of $f(x)$ would be lost, such as

$$\lim_{j\to-\infty} \|P_j f(x)\| = 0 \tag{36}$$

And considering Eq. (11), we have

$$P_{j+1} f(x) = P_j f(x) \dotplus g_j(x), \tag{37}$$

where

$$P_j f(x) = \sum_k c_{j,k}\phi(2^j x - k), \tag{38}$$

$$g_j(x) = \sum_k d_{j,k}\psi(2^j x - k), \tag{39}$$

Let $\mathbf{c}_J$ denote $\{c_{j,k}\}$, and $\mathbf{d}_J$ denote $\{d_{j,k}\}$. $\mathbf{c}_j$ and $\mathbf{d}_J$ mean the approximation and detail representations of $f(x)$, respectively. The decomposing process can be shown in Fig. 1.

Let the function of $f(x)$ be

$$f(x) = \sin(4\pi x^4). \tag{40}$$

Sampling it with the space 2^{-8}, we obtain the approximation c_0 of the function of $f(x)$ in scaling functions space $\boldsymbol{V}_0$. Consequently we suppose the detail d_0 is zero in wavelet functions space $\boldsymbol{W}_0$. Using 6-order Daubechies wavelet to decompose c_0, the approximation c_{-1} and detail d_{-1} at scale $j = -1$, as well as the approximation c_{-2} and detail d_{-2} at scale $j = -2$ are obtained, which are shown in Fig. 2.

From Fig. 2 we know that the discomposing detail d_j denotes the detail information of the function $f(x)$. As the scale decreases the detail information increases gradually; the larger the scale the lesser the details, so the approximating error becomes small.

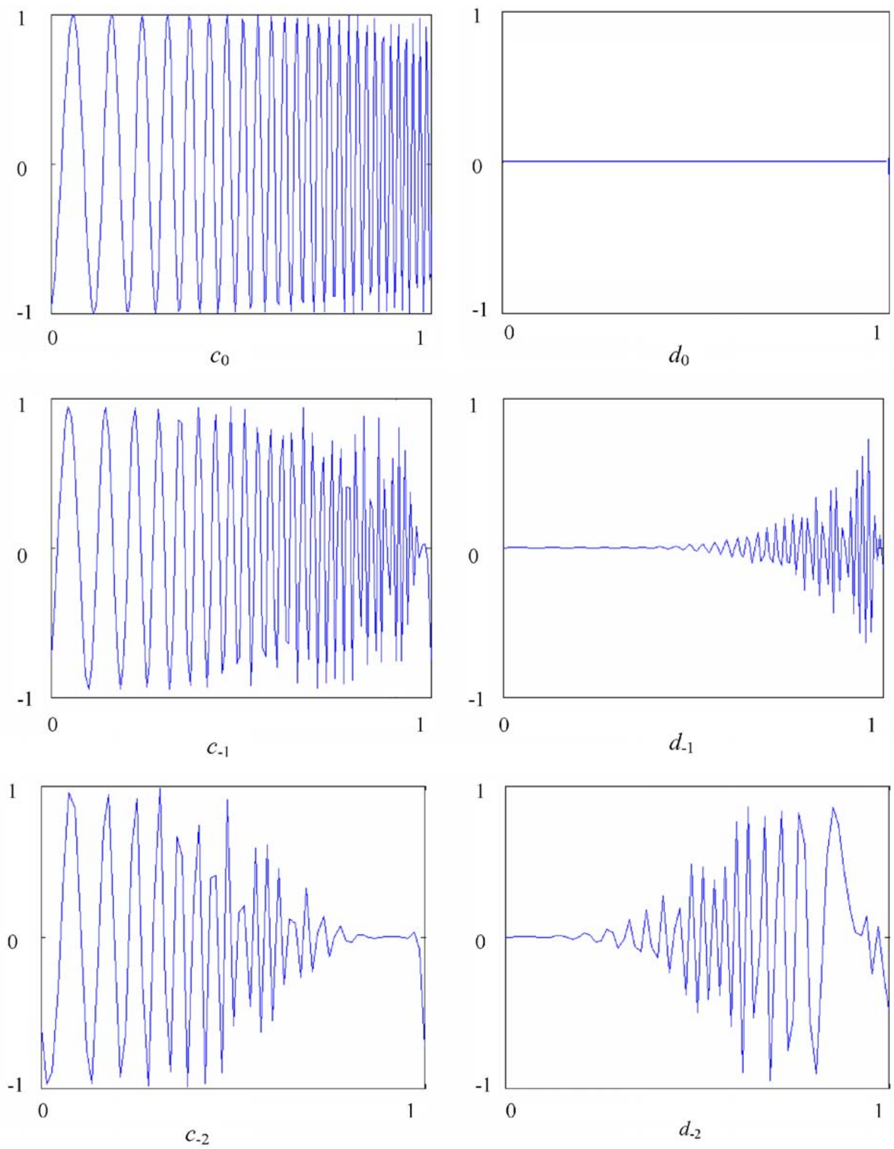

Fig. 2. The decomposition of function $f(x)$.

Considering the orthonormality of Daubechies wavelets, we have [14,11]

$$\|f\|_{L^2} = \left(\sum_{j=0}^{\infty} \|(P_{j+1} - P_j) f\|_{L^2}^2 \right)^{1/2} = \sum_{j=0}^{\infty} \|g_j(f)\|_{l^2}, \tag{41}$$

i.e., the L_2-norm of the function on the left equals the l_2-norm of wavelet coefficients on the right. So discarding small wavelet coefficients will change the function norm only by a same small amount. The

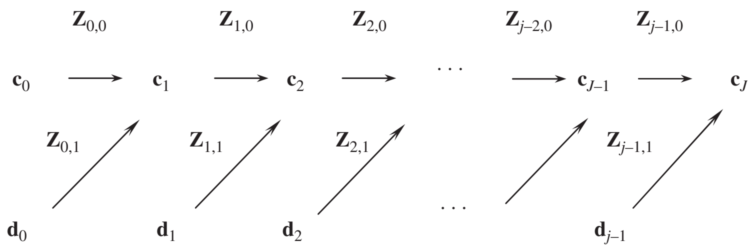

Fig. 3. Multiscale lifting scheme.

small digits will eventually have to become arbitrarily small, such as

$$\lim_{j\to+\infty} g_j(x) = 0. \tag{42}$$

As a result, every time the approximating wavelet representation $g_j(x)$ would be treated as an error estimator, which is capable of assessing the solution error. Let ε denote the total error bound, the rule to get finer solution is

$$\|g_j(x)\| \geqslant \varepsilon. \tag{43}$$

Using Eqs. (9) and (12), we have

$$\phi(2^j x - k) = \sum_s p_s \phi(2^{j+1}x - 2k - s), \tag{44}$$

$$\psi(2^j x - k) = \sum_s q_s \phi(2^{j+1}x - 2k - s). \tag{45}$$

Expressing in matrix form

$$\mathbf{\Phi}_j^{\mathrm{T}} = \mathbf{\Phi}_{j+1}^{\mathrm{T}} \mathbf{Z}_{j,0}, \tag{46}$$

$$\mathbf{\Psi}_j^{\mathrm{T}} = \mathbf{\Phi}_{j+1}^{\mathrm{T}} \mathbf{Z}_{j,1}, \tag{47}$$

where $\mathbf{\Phi}_j^{\mathrm{T}}$ denotes the vector $\{\phi_{j,k}(x)\}$, $\mathbf{\Phi}_j^{\mathrm{T}}$ denotes the vector $\{\phi_{j+1,2k+s}(x)\}$, the matrices $\mathbf{Z}_{j,0}$ and $\mathbf{Z}_{j,1}$ are formed with the consequences $\{p_s\}$ and $\{q_s\}$, and the superscript T denotes the transforming of matrix. Then we have [2]

$$\mathbf{\Phi}_j^{\mathrm{T}} \mathbf{c}_j + \mathbf{\Psi}_j^{\mathrm{T}} \mathbf{d}_j = \mathbf{\Phi}_{j+1}^{\mathrm{T}} (\mathbf{Z}_{j,0} c_j + \mathbf{Z}_{j,1} \mathbf{d}_j). \tag{48}$$

We can obtain the multiscale lifting process from lower space V_j to higher space V_{j+1}, as shown in Fig. 3.

The multiscale lifting algorithm is summarized as follows:

ALGORITHM

(1) *Construct initial meshes on* $\mathbf{\Omega}$, *select Daubechies wavelets of order N and approximate space* V_{j+1}, *and set constants* ε.

(2) *Compute the approximation*, *then determine whether it is a sufficiently good approximation or not according to Eq*. (43).
(3) *If* $\|g_j(x)\| \leqslant \varepsilon$, *stop and give the answer*; *else go the next step*.
(4) *According to Eq*. (48), *lift the approximate space and construct the new corresponding stiffness matrix*, *then go to the step* 2).

5. Numerical examples

In order to illustrate the effectiveness of such dynamic multiscale method for computation without involving the complexities caused by higher dimensions, first we give a wavelet-Galerkin scheme

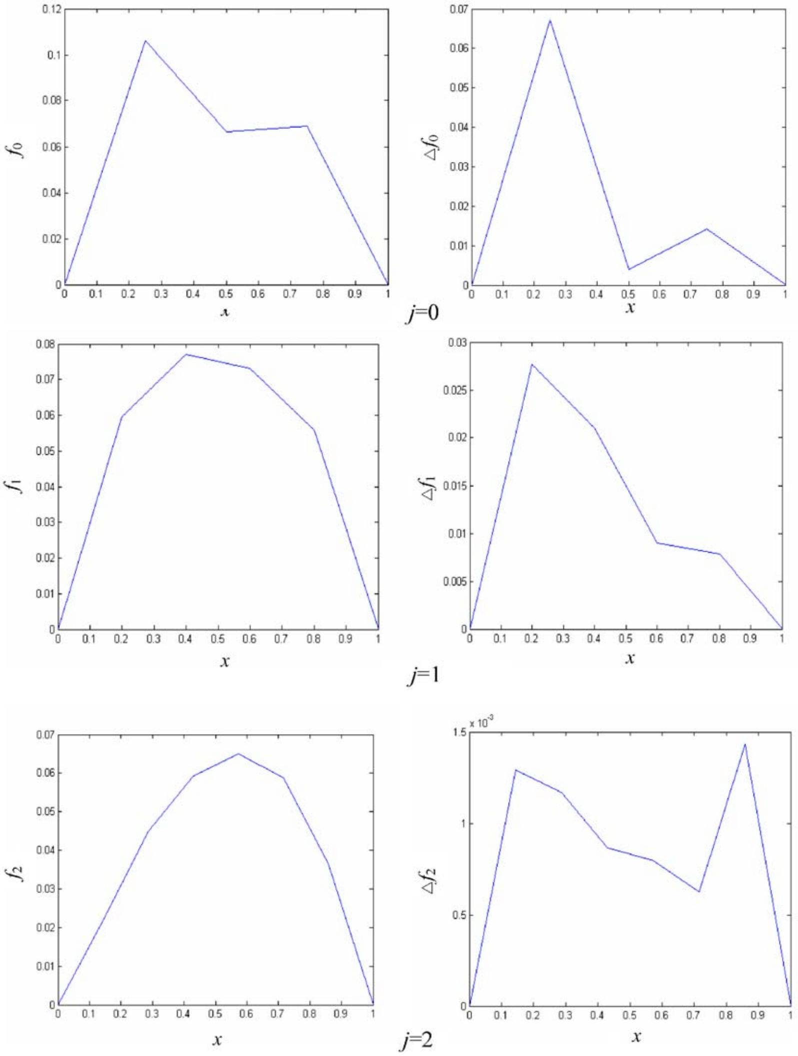

Fig. 4. Multiscale solution process.

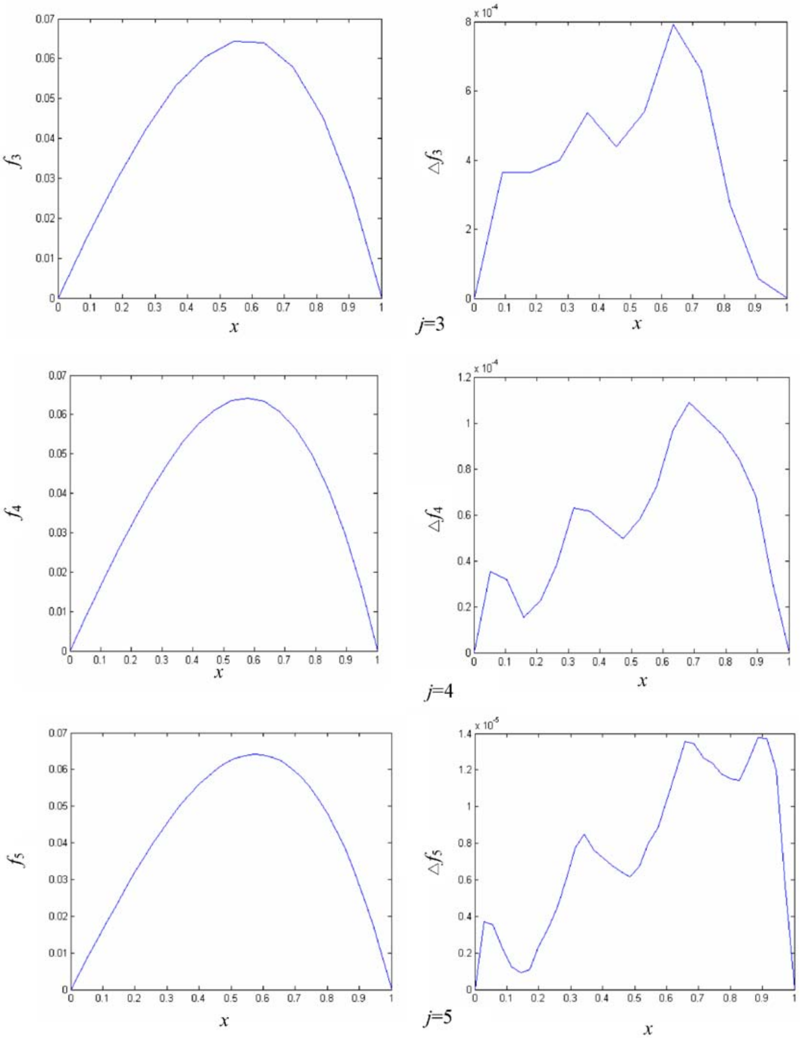

Fig. 4. (*continued*).

using the following one-dimensional differential equation:

$$
\begin{aligned}
&-f''(x) = x,\\
&f(x)|_{x=0,1} = 0.
\end{aligned}
\tag{49}
$$

Using the 3-order Daubechies wavelet-Galerkin method to solve it. Let the error bound be $\varepsilon = 2e-5$, the solving process is shown in Fig. 4. For different calculation scale j, the coordinate f_j denotes the solution result and Δf_j denotes the corresponding error. In the calculation the diagonal preconditioning [18,27] is usually used to avoid numerical instabilities and slow convergence for iterative resolution algorithms. The norm of relative error $\|\Delta f\|$ against the scale of wavelet is shown in Fig. 5.

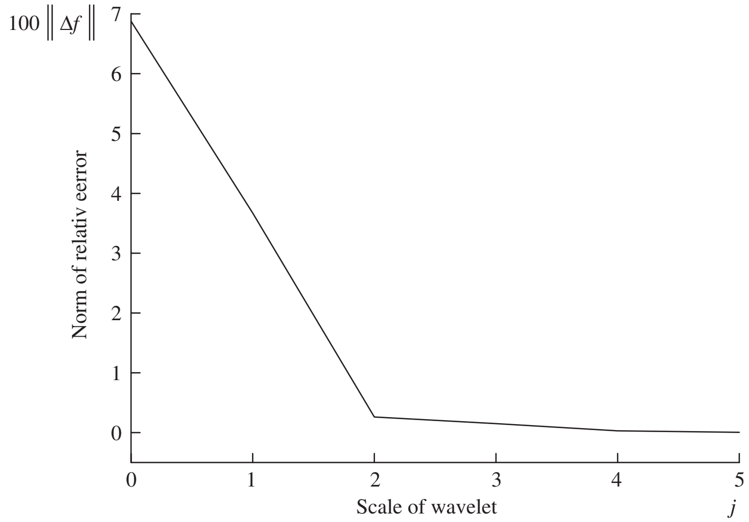

Fig. 5. Norm of relative error against scale of wavelet.

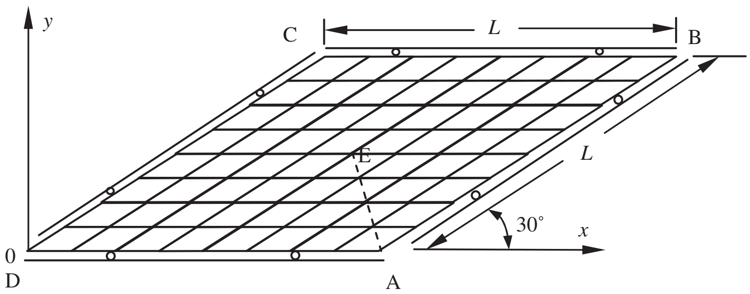

Fig. 6. Simply supported 30° skew plate.

From Fig. 4 we can see as the scale j increases, the approximation error to the function $f(x)$ becomes smaller. While the computation scale equals five, the solution error satisfies the bound requirement. In Fig. 5 the convergence is examined in terms of norm of relative error.

Another example is a skew plate. A 30° skew plate, simply supported on all edges with uniform load, is analyzed and a mesh division is illustrated in Fig. 6. The plate with side $L = 100$, thickness $t = 1.0$, Young's modulus $E = 10^6$ and Poisson rate $u = 0.3$, is loaded by a unit uniform loading. This problem is generally used to assess convergence properties of finite element method [19,29].

The deformation of the plate is solved using the traditional shell element [19] and the wavelet finite elements D30, D33, D35 [9], respectively. The brief introduction of eight-node quadrilateral isoparametric wavelet finite elements is listed in the appendix. The reference and results for transverse displacement $\boldsymbol{w}_j$ solved at scale j along the line A–E are shown in Fig. 7.

The horizontal coordinate r stands for the distance from A to E. The reference is obtained through the traditional shell element with 64×64 mesh, which has six degrees of freedom at each node: translations

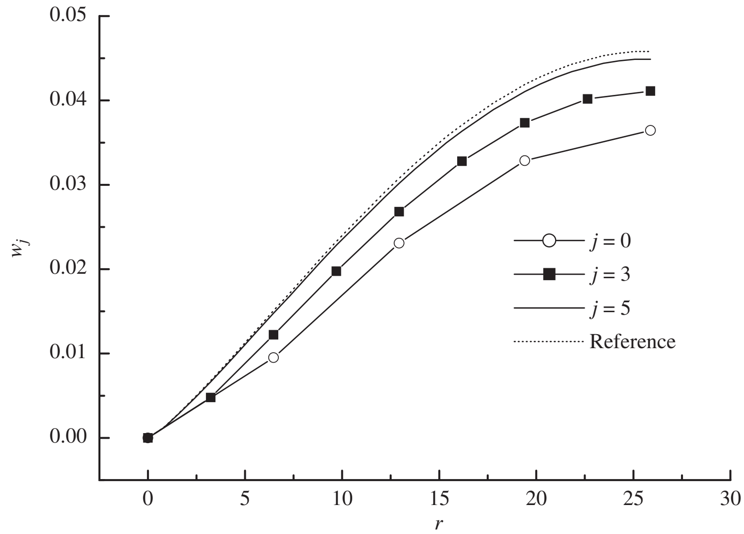

Fig. 7. Comparison of transverse displacement along the line A-E.

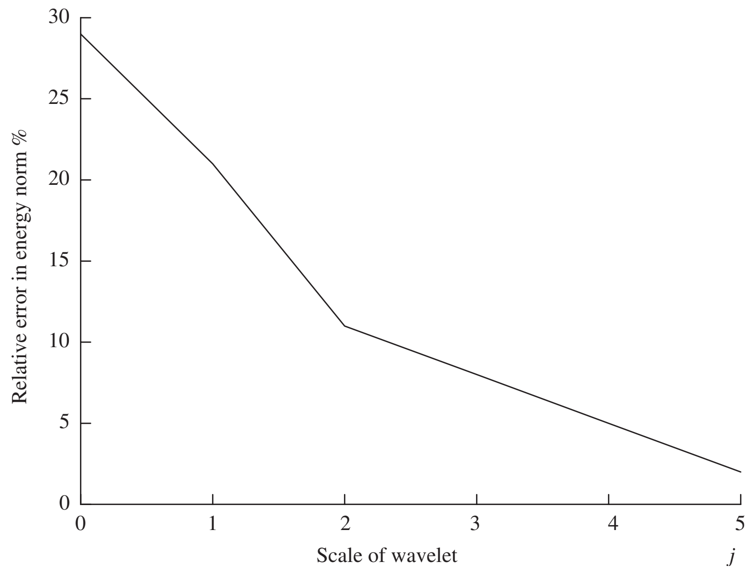

Fig. 8. Relative error in energy norm against scale of wavelet.

in the node x, y and z directions and rotations about the nodal x, y and z-axes. The results are obtained through wavelet finite elements D3j, which denote 3-order wavelets at scale j are used as interpolation functions, with 8×8 mesh in Fig. 6. From Fig. 7, it can be seen that the relative errors decrease very rapidly with the scale j becoming larger. Convergence in the energy norm is also examined from Fig. 8.

6. Conclusion

Wavelet multiresolution analysis provides a powerful framework for analyzing functions at various scales. Due to the fact that Daubechies wavelets possess several properties, such as compact support, vanishing moments and orthogonality, it has gained great interest in numerical analysis, especially in singular problem [22], which leads to the problem of computing integrals of products of derivatives of wavelets. By using the two-scale equation and inhomogeneous conditions arising from moment equations satisfied by the wavelet, the method to calculate multiscale connection coefficients for stiffness matrices and load vectors is presented for the first time. Based on such a method, the algorithm of multiscale lifting computation is developed. The validity of the proposed method and the effectiveness of the multiscale lifting scheme are demonstrated by two numerical examples.

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Appendix A

A.1. Some properties of Daubechies wavelets

A.1.1. Compact support

A function has compact support if it is identically zero outside a finite interval. The scaling function $\phi_N(x)$ and wavelet function $\psi_N(x)$ of Daubechies wavelets have finite length, where N denotes the order of wavelets. Corresponding support intervals are

$$\operatorname{supp}\phi_N = [0, 2N-1], \tag{A.1}$$

$$\operatorname{supp}\psi_N = [1-N, N]. \tag{A.2}$$

A.1.2. Vanishing moments

Daubechies wavelet with order N is orthogonal to polynomials up to $N-1$ order. The wavelet function $\psi_N(x)$ is said to have $N-1$ order of vanishing moments

$$\int_{-\infty}^{\infty} x^k \psi_N(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, N-1. \tag{A.3}$$

A.1.3. Orthogonal property

The scaling function $\phi_N(x)$ and wavelet function $\psi_N(x)$ satisfy the following orthogonal conditions:

$$\langle \psi_{j,k}(x), \quad \psi_{l,m}(x)\rangle = \delta_{j,l}\delta_{k,m}, \tag{A.4}$$

$$\langle \phi_{j,k}(x), \quad \phi_{l,m}(x)\rangle = \delta_{j,l}\delta_{k,m}, \tag{A.5}$$

$$\langle \phi_{j,k}(x), \quad \psi_{l,m}(x)\rangle = 0, \tag{A.6}$$

where $j, k, l, m \in Z$, $\delta_{j,l} = \begin{cases} 1 & j = l, \\ 0 & j \neq l, \end{cases} \quad \delta_{k,m} = \begin{cases} 1 & k = m, \\ 0 & k \neq m. \end{cases}$.

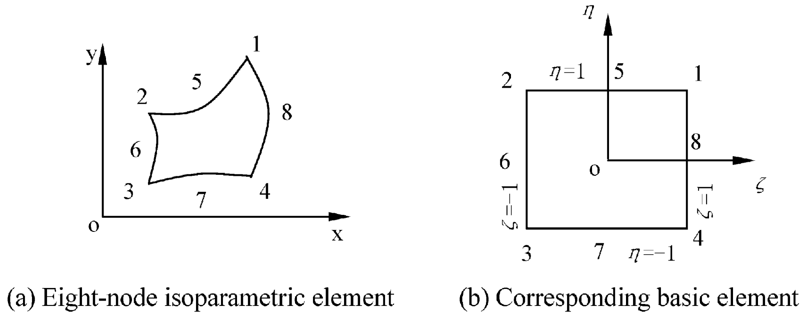

Fig. 9. Wavelet finite element: (a) Eight-node isoparametric element and (b) corresponding basic element.

A.2. Eight-node quadrilateral isoparametric wavelet finite elements

The eight-node quadrilateral isoparametric element and the corresponding basic element are shown in Fig. 9.

The degree of freedom of the element is

$$\mathbf{a} = [\mathbf{a}_1^{\mathrm{T}} \quad \mathbf{a}_2^{\mathrm{T}} \quad \mathbf{a}_3^{\mathrm{T}} \quad \mathbf{a}_4^{\mathrm{T}} \quad w_5 \quad w_6 \quad w_7 \quad w_8]^{\mathrm{T}}, \tag{A.7}$$

where w is the nodal displacement, and

$$\mathbf{a}_i = \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \begin{Bmatrix} w_i \\ \dfrac{\partial w_i}{\partial y} \\ \dfrac{\partial w_i}{\partial x} \end{Bmatrix} \quad (i = 1, 2, 3, 4) \tag{A.8}$$

Then the stiffness matrix in wavelet space is

$$\begin{aligned}\tilde{\mathbf{K}} = D_0(&\mathbf{\Lambda}_N^{j,0,0}(x) \otimes \mathbf{\Lambda}_N^{j,2,2}(x) + v((\mathbf{\Lambda}_N^{j,0,2}(y))^{\mathrm{T}} \otimes \mathbf{\Lambda}_N^{j,0,2}(x) + \mathbf{\Lambda}_N^{j,0,2}(y) \otimes (\mathbf{\Lambda}_N^{j,0,2}(x))^{\mathrm{T}}) \\ &+ \mathbf{\Lambda}_N^{j,2,2}(y) \otimes \mathbf{\Lambda}_N^{j,0,0}(x) + 2(1-v)\mathbf{\Lambda}_N^{j,1,1}(y) \otimes \mathbf{\Lambda}_N^{j,1,1}(x))\end{aligned} \tag{A.9}$$

where D_0 is the bending rigidity and v is Poisson's ratio.

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